ANALYSIS OF MARKET COMPETITION

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Abstract. This project explores the application of ordinary differential equations to model market share dynamics. The discussion will follow the translation of real-world factors such as product quality, marketing prowess, brand loyalty, and market competition into mathematical inputs for the proposed ODE. While acknowledging the limitations of the simplifying assumptions, the study highlights the great potential of ODE-based modeling as a powerful tool for analyzing general market trends and phenomena.

1. Background/Motivation

The study of market dynamics often involves understanding how competing products or services gain, lose, or sustain market share over time. In this project we choose to analyze the relative market share of companies in a particular cross-section of the market. In particular, we chose to analyze the carbonated beverage industry, but similar techniques could be easily adapted to analyze other sectors of the market that are reasonably isolated. To employ the use of ODEs, we make the simplifying assumption that the relative market share of company i is only a function of time, allowing us to capture the temporal evolution of market shares. We acknowledge that relative market shares are in fact complicated functions of marketing strategies, product innovation, leadership, etc. and do our best to incorporate such information into the model as parameters.

This model is motivated by the desire to simplify complex dynamics, guide strategic decision-making, and offer predictive insights in competitive markets. The impact of this model lies in its ability to translate abstract business dynamics into actionable, data-driven conclusions.

1.1. Related Work.

A similar adaptation of the Lotka-Volterra model to what we propose is given in [MPR16]. This paper and others only briefly describe and all go beyond the Lotka-Volterra model into various adaptations. Unlike other research done we focus our study primarily on the Lotka-Volterra model, how it models various market events, and it's limitations. For other evidence of how systems of ODEs have proven their utility in analyzing time-dependent market phenomena, we refer you to [Bar19, CTS$^+$21, Rom13].

Date: 4 October 2024.

2. Modeling

2.1. Simplifying Assumptions. We begin by making very clear the simplifying assumptions that are made.

- **A single product market**: Most companies sell multiple products in various markets, but we will assume that we are dealing with a market that only has one product. The running example that you will see discussed in this paper is the industry for carbonated beverages.
- **No macroeconomic factors**: In reality, every company not only depends on its competitors in their narrow slice of the total market, but also the macro economy of the nation/globe. These factors will not be considered in this model.
- **All companies represented make up the full market share**: The size of an industry along with the number of companies is always growing or shrinking. In some cases there are even untapped markets with no company servicing those customers. Our model assumes that the market is composed of only the companies represented by the ODE and that there are no untapped customers.

Note that all values in our model are represented as proportions and are dimensionless with the exception of the alpha and beta values which have dimension $[T^{-1}]$, the same as the derivative $\frac{dX_i}{dt}$.

2.2. Base Model. The standard ODE for modeling competing entities are the Lotka-Volterra equations. A standard Lotka-Volterra model considers each entity to be either a predator or a prey and does not have the ability to model an entity which is both predator and prey simultaneously. When considering the problem of modeling multiple companies competing in a specific cross section of a market every company can be considered both predator and prey at any point of time. In order to adapt the Lotka-Volterra model to these needs, the following n equations were derived for $i \in \{1, 2, \ldots, n\}$ where n is the number of companies being considered.

$$\frac{dX_i}{dt} = X_i \left(\alpha_i \left(1 - \frac{X_i}{K} \right) + \sum_{j \neq i} \beta_{ij} X_j \right) \tag{1}$$

The parameters are described in the following way.

- $X_i \in \mathbb{R}$: the percentage of the market that company i controls. As the above model does not inherently enforce the constraint that $\sum_i X_i = 1$, we artificially impose this by normalizing numerical solutions. We recognize this is a potential weakness, for more on this see 2.3.
- $\alpha_i \in \mathbb{R}$: the specific rate of growth (or decline) for company i. This rate is dependent on numerous factors of business quality such as a company i's product, customer service, supply chain, or advertising.

α_i is also dependent on growth of financial metrics such as revenue, profitability, and cash flow.
- $\beta_{ij} \in \mathbb{R}$: the level of competition between companies X_i and X_j. A negative value indicates that i is winning market share at the expense of company j. A positive number indicates the competition is enhancing rather than inhibiting to market share growth, which would imply other companies would lose market share to account for i and j advancing.
- $K = \frac{2}{3}$: the carrying capacity for each company. In theory, a strong company would be able to assert control over the entire market, but the US D.O.J protects against this. According to their website [oJ08], the largest market share that a company can control is our assigned value of $\frac{2}{3}$ and any company possessing more than $\frac{2}{3}$ of the market would subsequently be broken into smaller companies.

2.3. **Previous Model Iterations.** Initially, we were concerned that our base model does not naturally enforce the constraint $\sum_i X_i = 1$. Normalizing the market shares imposes this constraint but distorts the relative dynamics between companies. To address this, we considered reformulating the system as a differential-algebraic equation (DAE):

$$\frac{dX_i}{dt} = X_i \left(\alpha_i \left(1 - \frac{X_i}{K} \right) + \sum_{j \neq i} \beta_{ij} X_j \right), \quad \text{s.t.} \ \sum_i X_i = 1.$$

However, numerical methods struggled with this approach. Alternatively, we experimented with projecting X_i onto the simplex $\sum_i X_i = 1$ at each step, but this frequently reduced market shares to just one or two dominant companies, which did not align with our understanding of market behavior.

Ultimately, we opted to apply artificial normalization, recognizing it as a limitation of our model. Note that we chose not to include visualizations of previous models in this report due to a lack of space. The interested reader can consult the attached Jupyter notebook.

2.4. **Stability Analysis.** For given values of α, β, it is a relatively straightforward process to analyze the stability of various equilibrium points of the system.

2.4.1. *Reasonable Parameter Values.* With a little bit of data analysis and a few simplifications of our model, we found viable constants α, β that enables for the exploration of stability. Assume now that our model only consists of the companies CocaCola, PepsiCo, and Other (here, 'Other' denotes the sum of all other carbonated beverage companies). Looking at market data for these companies from 2017-2023 [Cen24], we estimated average growth and competition rates between the companies as

$$\alpha = \begin{bmatrix} 0.024 & 0.0185 & 0.0047 \end{bmatrix} \quad \beta = \begin{bmatrix} 0 & -0.0064 & -0.0041 \\ -0.0064 & 0 & -0.0026 \\ -0.0041 & -0.0026 & 0 \end{bmatrix}$$

For a more in-depth discussion of how these parameters were approximated, see section 3.1

2.4.2. *Equilibria and Stability.* We can linearize the system 1, i.e. compute $Df(\mathbf{X})$ adapted to the case of $n = 3$ as

$$\begin{bmatrix} A_1 & \beta_{12}X_1 & \beta_{13}X_1 \\ \beta_{21}X_2 & A_2 & \beta_{23}X_2 \\ \beta_{31}X_3 & \beta_{32}X_3 & A_3 \end{bmatrix}, \quad \text{where } A_i = \alpha_i - \frac{2\alpha_i X_i}{K} + \sum_{j \neq i} \beta_{ij} X_j$$

using numerical solvers (we used Sympy and Scipy), we can compute and classify the following 8 equilibrium points.

X_1	X_2	X_3	Stability
0.00	0.00	0.00	undetermined
0.00	0.00	0.67	unstable
0.00	0.63	0.44	unstable
0.00	0.67	0.00	unstable
0.56	0.52	0.15	stable
0.57	0.53	0.00	unstable
0.63	0.00	0.30	unstable
0.67	0.00	0.00	unstable

In this case, $\sum_i X_i \neq 1$, as we analyze the model in its raw, non-normalized form. While the equilibrium points are valid for studying the system's dynamics, their proportions should be interpreted cautiously since they do not represent normalized probabilities or shares. Notably, the only stable equilibrium occurs when X_1, X_2, and X_3 are all nonzero, suggesting the model inherently favors coexistence between companies, as observed in reality. A detailed bifurcation analysis of this phenomenon would be an intriguing avenue for future work.

2.5. **Incorporating Market Shocks.** It is also of interest to consider the dynamics when a shock is introduced to the market. Such shocks include, but are not limited to, the introduction of a new product, a merger/acquisition between companies, or the introduction of a new company. Each of these shocks can be integrated into the model without too much difficulty. In this section, we propose mathematical formulations of various shocks. For visualizations of simulations for each of these market shocks, refer to 3.2.

2.5.1. *Introduction of a New Company.* It is not uncommon for new beverage companies to enter the market and cause disruptions (for example, caffeinated chocolate milk?). Assume that we know that a company X_6, will enter the market at time $t = t_*$. We can then introduce the following term to our model (keeping the same equations for $X_1, \dots, X_5$):

$$\frac{dX_6}{dt} = \mathbb{1}_{[t_*,\infty)}(t) X_6 \left(\alpha_6 \left(1 - \frac{X_6}{K} \right) + \sum_{j \neq i} \beta_{ij} X_j \right) \tag{2}$$

This equation introduces a new competitor to the system, whose market share X_6 remains constant at zero until the company enters the market at time $t = t_*$, after which it evolves dynamically.

2.5.2. *Introduction of a Popular New Product.* Consider the case of when a popular new product is introduced by a company. We make the simplifying assumption that this will increase only the growth rate of the company that creates the new product, the remaining parameters will stay the same. Suppose that at time $t = t_*$ company i introduces a popular new product. If initially company i had growth rate $\alpha_i = \alpha_i^{(1)}$, then we assume that after time t_*, the company now has growth rate $\alpha_i^{(2)}$, where $\alpha_i^{(2)} \geq \alpha_i^{(1)}$. That is, we adjust

$$\alpha_i(t) = \begin{cases} \alpha_i^{(1)} & \text{if } t < t_* \\ \alpha_i^{(2)} & \text{if } t \geq t_* \end{cases} \tag{3}$$

2.5.3. *Merger/Acquisition.* Although mergers and acquisitions are technically two different business actions, we will model them as the same. Without loss of generality, suppose that companies 4 and 5 merge at time t_*. After the event, we may (again, WLOG) assume that X_4 is the remaining company; that is, company 5 was absorbed into company 4. This enables us to write

$$\begin{bmatrix} \frac{dX_1}{dt} \\ \vdots \\ \frac{dX_4}{dt} \\ \frac{dX_5}{dt} \end{bmatrix} = \begin{cases} \begin{bmatrix} X_1 \left(\alpha_1 \left(1 - \frac{X_1}{K}\right) + \sum_{j=2}^{5} \beta_{1j} X_j \right) \\ \vdots \\ X_5 \left(\alpha_5 \left(1 - \frac{X_5}{K}\right) + \sum_{j=1}^{4} \beta_{5j} X_j \right) \end{bmatrix}, & \text{if } t < t_*, \\ \\ \begin{bmatrix} X_1 \left(\alpha_1 \left(1 - \frac{X_1}{K}\right) + \sum_{j=2}^{4} \overline{\beta}_{ij} X_j \right) \\ \vdots \\ X_4 \left(\overline{\alpha}_4 \left(1 - \frac{X_4}{K}\right) + \sum_{j=1}^{3} \overline{\beta}_{ij} X_j \right) \\ 0 \end{bmatrix}, & \text{if } t \geq t_*. \end{cases} \tag{4}$$

Notice here that the growth rate α has only changed for company 4. Competition rates β may or may not have changed, but we leave that up to the reader's discretion in their endeavors as how these rates change must be considered on a case-by-case basis.

2.5.4. *Disastrous Event for One Company.* No company is immune to disasters; examples of such could be legal issues, management changes, or product failures. We model such events as a sharp decrease in the growth rate α_i at time t_d, followed by a gradual recovery to its initial value over a specified interval. Specifically, at t_d, α_i drops from $\alpha_i^{(1)}$ to $\alpha_i^{(2)}$, where $\alpha_i^{(2)} \ll \alpha_i^{(1)}$. Over the recovery period $[t_d, t_r]$, α_i increases linearly back to $\alpha_i^{(1)}$. We can write this as

$$\alpha_i(t) = \begin{cases} \alpha_i^{(1)}, & t < t_d, \\ \alpha_i^{(2)} + \left(\alpha_i^{(1)} - \alpha_i^{(2)}\right) \cdot \frac{t-t_d}{t_r-t_d}, & t_d \leq t \leq t_r, \\ \alpha_i^{(1)}, & t > t_r. \end{cases} \tag{5}$$

There are, of course, many other ways to model this, as well as all of the previous market shock considerations.

3. Results

3.1. **Base Model Against Historical Data.**

While the proposed model seems to theoretically account for the main drivers in market share expansion, we endeavor to examine the validity of the model when applied to real-world market dynamics. Figure 1 displays the relative market share among four of the largest carbonated beverage companies in the United States for the years 2012-2023 [Cen24].

The most difficult aspect of replicating similar dynamics to Figure 1 was estimating the correct α and β values. This estimation, using financial metrics like growth in return on assets, profitability, and free cash flow, was ineffective because while these provide economic rationale for market share expansion, they offer no insight into how to merge such metrics effectively.

In contrast to using company financial data, technical indicators were devised from the relative market share data itself using functions from the Pandas library. Values of α for each company are calculated at each time t as an exponential weighted moving average with a window of six months. The beta vector is correspondingly calculated as the six month rolling correlation between the given company and the other companies in the competition space. Thus, α and β are time-dependent functions of the previous market share values. This is a modification to our original model (where we assumed constant α, β) which we feel is beneficial to replicate realistic dynamics.

For simplicity, we model three companies, Coca-Cola, Pepsi, and Other, with 'Other' being a representation of the rest of the beverage industry. In calculating the α and β values, scaling was performed to obtain stable solutions. The resulting model is visualized in Figure 2 ([Cen24]).

As a final step to our testing, we examined the model's utility in projecting future market share values. As alpha and beta are dependent upon

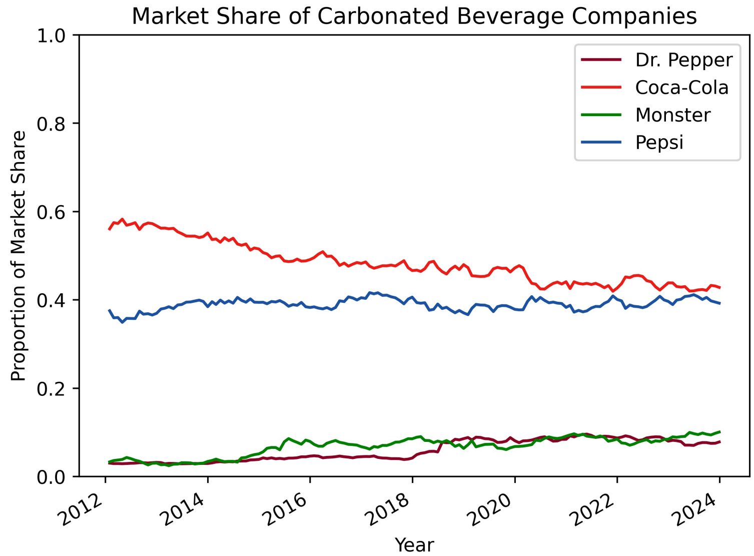


FIGURE 1. Relative market shares of 5 chosen carbonated beverage companies (Coca-Cola, Pepsi, Dr Pepper, Celsius, and Monster) from the years 2012-2024.

ongoing data in the previous example, extending the model into the future fails. To account for this limitation, the model was modified such that initial alpha and beta metrics were created and then held constant. Figure 3 demonstrates the projected equilibrium of this system. It may be observed that Coke and Pepsi lose market share to new challengers. This presents possible future areas of research, particularly in conducting investment analysis.

3.2. **Incorporating Market Shocks.** For each of the shocks presented in the modeling section, we present figures and interpretations.

3.2.1. *Introduction of a Popular New Company 4a.* When a strong competitor enters the market with a unique product or way of marketing, such as viral backing or influencer endorsements, it can significantly disrupt the dynamics of the industry. As shown above, the relative market share of all existing companies declined at a similar rate, indicating that they collectively suffered losses due to the new entrant's presence. Such events are becoming increasingly common with the rise of social media influencers and the potential of viral videos and trends to rapidly amplify a brand's visibility and appeal.

3.2.2. *Introduction of a New Product 4b.* In a competitive market, companies strive to constantly improve their products. Occasionally, a company will come out with a new and revolutionary product which will create a

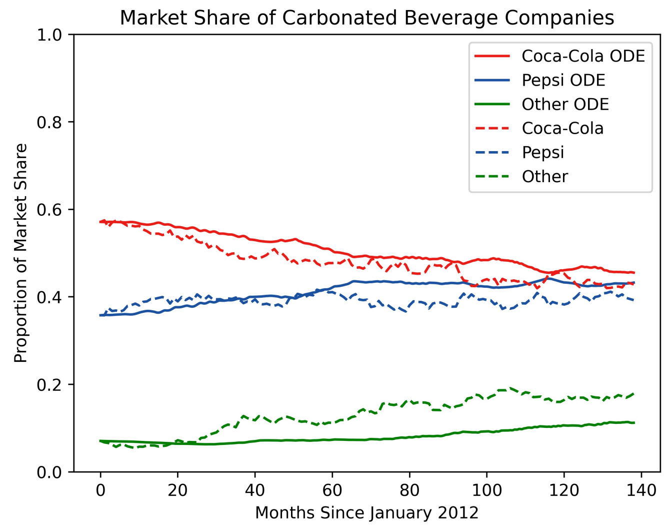


FIGURE 2. A comparison of our proposed ODE system with real historical data. Values of α and β were approximated with the historical data at each time step.

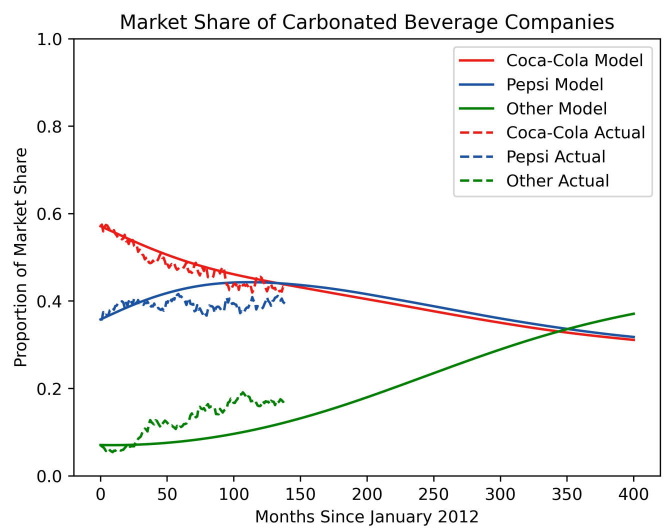
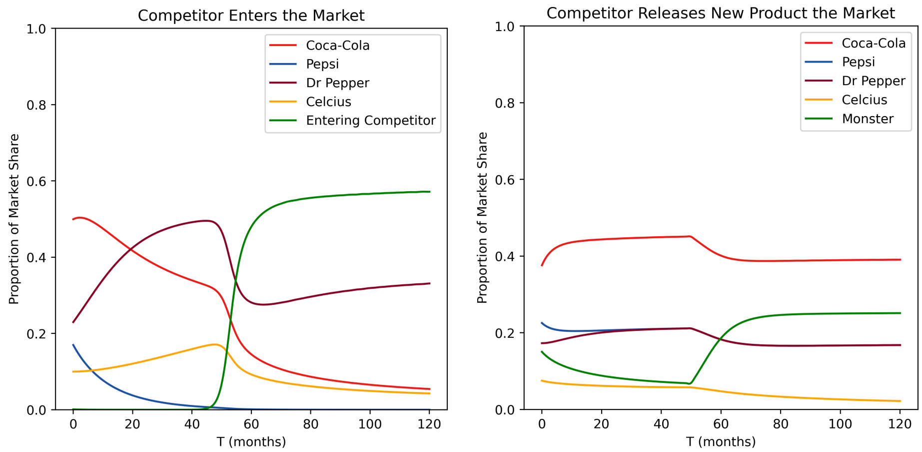


FIGURE 3. Application of our system of ODEs to project future market dynamics. Initial values for α and β are based on historic data, then held constant through all time steps.

market shock. A potential example of this is the release of the iPhone in 2007 by Apple which caused a rival phone producer, BlackBerry to crumble.



(A) (B)

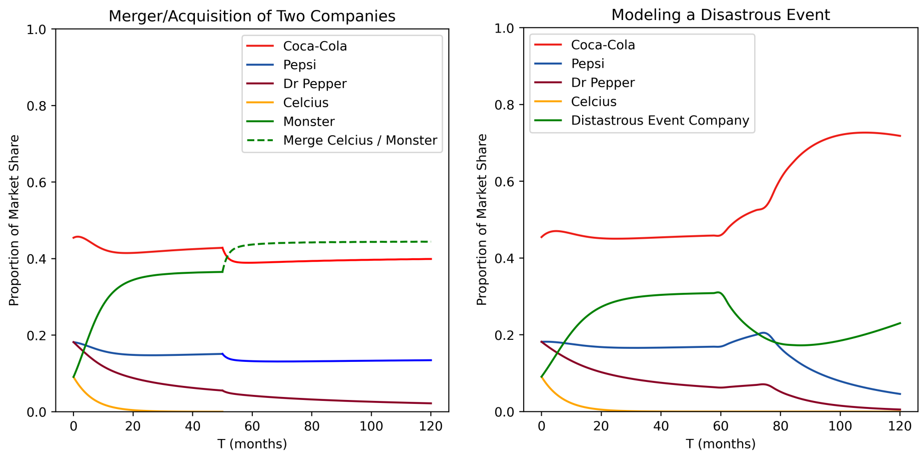
FIGURE 4. (A) The entering competitor's (green) growth rate is switched on at time $t \approx 50$. (B) Monster (green) introduces a new popular product at time $t \approx 50$, modeled with a sudden spike in its growth rate.

3.2.3. *Merger/Acquisition 5a.* Mergers are a frequent occurrence in the corporate world, often serving as a strategic tool for growth and expansion. In such events, a strong company may acquire a struggling one, leveraging its assets, rights, and inventory to capitalize on market opportunities. This strategy allows the acquirer company to improve its reach and net worth while revitalizing underutilized resources. The impact of such mergers often reflects a shift in market dynamics, with changes proportional to the size of the companies involved. This phenomenon is both fascinating and widely observed across various market sectors.

3.2.4. *Disastrous Event for One Company 5b.* When a company experiences significant losses due to a public incident, such as a scandal or sudden negative news, its market share is likely to decline, at least temporarily. This drop can result from negative public opinion, financial or operational setbacks, or declining stock prices. In the model shown, the company's growth rate was sharply reduced for a period, reflecting the time it may take to recover from such an event, which could span months or even years. Meanwhile, the other companies in the market experienced slight increases in their market share, highlighting how such incidents can create opportunities for competitors to gain an advantage.

4. Analysis/Conclusions

Throughout the process, we used a variety of techniques to model relative market share, each of which yielded different levels of success. We measure



(A) (B)

FIGURE 5. (A) We modeled Monster (green) merging with Celsius (yellow). This was done by merging values for the two companies at time $t \approx 50$ and then treating the combined company as Monster with a higher growth rate. (B) A certain company (green) undergoes a disastrous event at time $t \approx 60$. This was modeled as their growth rate suddenly shrinking before linearly increasing to the original value.

success through the ability of our model to fit the real-world data as shown above. Ultimately, our final model effectively captured real-world data from recent years, indicating success. However, given more time, we would focus on enhancing the system's stability, refining key metrics for alpha and beta, and implementing a cosine scheduler to fine-tune the model. These improvements would contribute to a more robust model, providing more detailed and accurate results.

One key takeaway from the project was the importance of time-varying parameters in modeling complex phenomena. Models with fixed parameters often struggle to replicate dynamic market behavior effectively.

The models that performed successfully were able to replicate past data and project future trends. This capability is invaluable for companies looking to forecast their market share and make informed predictions about future developments. Although it remains challenging to account for unpredictable market events, companies can still leverage the model by incorporating factors such as new product launches, leadership changes, or the addition of third-party consultants. This approach can offer valuable information, guiding leaders in making decisions that drive future growth while also providing a clearer understanding of competitors.

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